Complete and Bipartite Graphs

Certain types of graphs play prominent roles in graph theory. A complete graph is a simple graph in which any two vertices are adjacent. A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y ; such a partition (X, Y) is called a bipartition of the graph, and X and Y its parts. We denote a bipartite graph G with bipartition (X, Y) by $G[X, Y]$. If $G[X, Y]$ is simple and every vertex in X is joined to every vertex in, then G is called a complete bipartite graph. Y: If $G[X, Y]$ is complete with $|X| = r$ and $|Y| = s$; then $G[X, Y]$ is denoted by $K_{r,s}$. A star is a complete bipartite graph $G[X, Y]$ with $|X| = 1$ or $|Y| = 1$.

Definition

A complete graph is a graph with n vertices and an edge between every two vertices.

- There are no loops.
- Every two vertices share exactly one edge.

We use the symbol K_n for a complete graph with n vertices.

The Properties of the complete graph K_n

- Each vertex has degree $n - 1$.
- The sum of all degrees is $n(n - 1)$.
- The number of edges in K_n is $\frac{n(n-1)}{2}$.

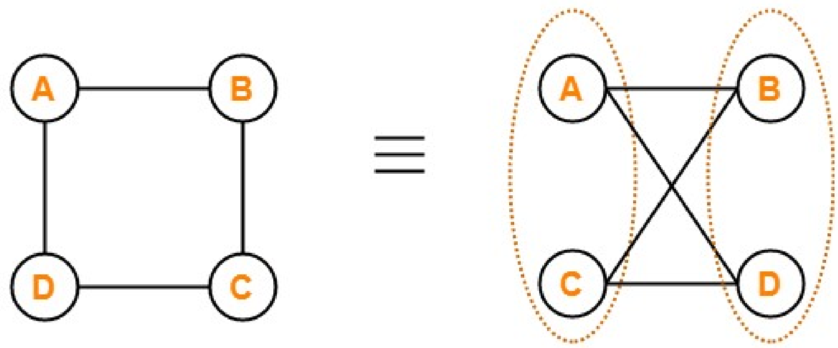
Complete bipartite graph

A bipartite graph is a special kind of graph with the following properties-

- It consists of two sets of vertices X and Y.
- The vertices of set X join only with the vertices of set Y.
- The vertices within the same set do not join.

Examples

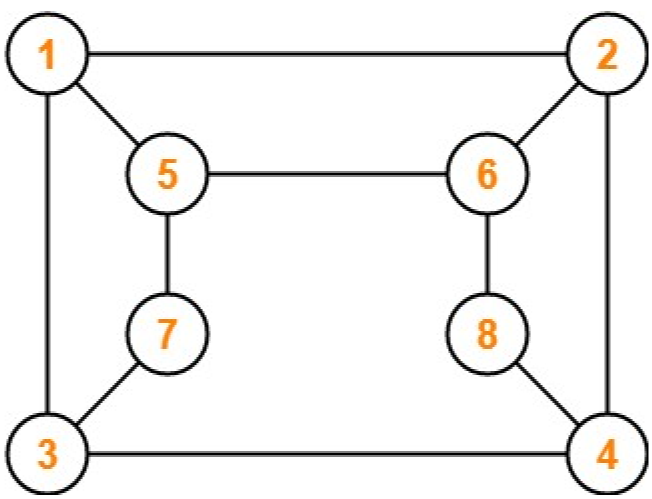
1.



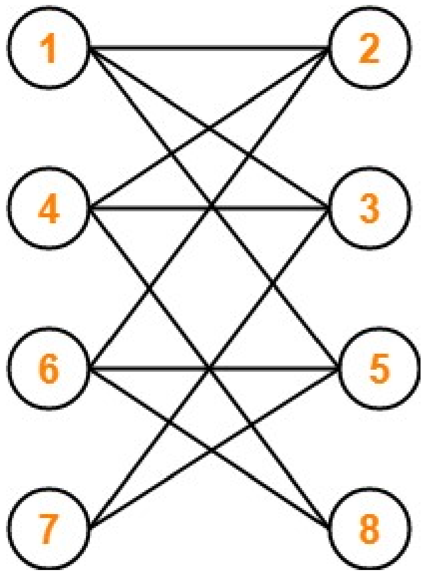
Here,

- ❖ The vertices of the graph can be decomposed into two sets.
- ❖ The two sets are $X = \{A, C\}$ and $Y = \{B, D\}$.
- ❖ The vertices of set X join only with the vertices of set Y and vice-versa.
- ❖ The vertices within the same set do not join.
- ❖ Therefore, it is a bipartite graph.

2. Verify where the following graph a bipartite graph?



The given graph may be redrawn as-



Here,

- ❖ This graph consists of two sets of vertices.
- ❖ The two sets are $X = \{1, 4, 6, 7\}$ and $Y = \{2, 3, 5, 8\}$.
- ❖ The vertices of set X are joined only with the vertices of set Y and vice-versa.
- ❖ Also, any two vertices within the same set are not joined.

This satisfies the definition of a bipartite graph. Therefore, a given graph is a bipartite graph.

Properties of a bipartite graph

- ❖ Bipartite graphs contain no odd cycles.
- ❖ Every subgraph of a bipartite graph is itself bipartite.
- ❖ In any bipartite graph with bipartition X and Y, Sum of the degree of vertices of set X = Sum of the degree of vertices of set Y.

Result

X If G is a bipartite graph and the bipartition of G is X and Y, then $\sum_{v\in X} \deg(v) = \sum_{v\in Y} \deg(v)$.

Proof

By induction on the number of edges. We denote the number of elements of X as |X|. Suppose |X| = r and |Y | = s for some integers r, s > 1. Note that the case where X or Y has one vertex is trivial, as only one edge can be drawn. Take the subgraph of G which consists of only the vertices of G. Now we begin inducting: add one edge from any vertex in X to any vertex in Y. Then, $\sum_{v\in X} \deg(v) = \sum_{v\in Y} \deg(v) = 1$. Now, suppose this is true for n-1 edges and add one more edge. Since this edge adds exactly 1 to both $\sum_{v\in X} \deg(v)$ and $\sum_{v\in Y} \deg(v)$., we have that this is true for all n∈N.

Theorem

A nontrivial graph G is a bipartite graph if and only if G contains no odd cycles.

Proof

If G is bipartite with bipartition X, Y of the vertices, then any cycle C has vertices that must alternately be in X and Y. Thus, since a cycle is closed, C must have an even number of vertices and hence is an even cycle. First, we prove the result in the case where G is connected. Assume G has no odd cycle. Let $u \in V(G)$ be an arbitrary vertex. Partition all other vertices based on the parity of distance (even or odd) from u.

Note that If no u, v –path exists, we say that $d(u, v) = \infty$ (or sometimes it is said to be undefined). That is, let $X = \{v \in V(G) : d(u, v)$ is even$\}$, $Y = \{v \in V(G) : d(u, v)$ is odd$\}$. Clearly, $X \cap Y = \emptyset$ and $X \cup Y = V(G)$ since G is connected. We claim that X, Y is a bipartition of G. Suppose not – then there exists an edge incident to two vertices of X or an edge incident to two vertices of Y. Without loss of generality, assume the former. Let $x_1, x_2 \in X$ and $x_1 \sim x_2$. It follows that $x_1 \in X \Rightarrow \exists\, u, x_1$-path P_1 of even length, $x_2 \in X \Rightarrow \exists\, u, x_2$-path P_2 of even length. Concatenate u, x_1-path P_1, the edge x_1x_2, and the x_2, u-path $P_2{}^{-1}$ to obtain a closed odd walk. If a graph contains an odd closed walk, then it contains an odd cycle. So the graph must contain an odd cycle. ⇒⇐ Hence, it must be the case the X, Y is a valid bipartition, so G is bipartite.

Complete Graph

A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G:

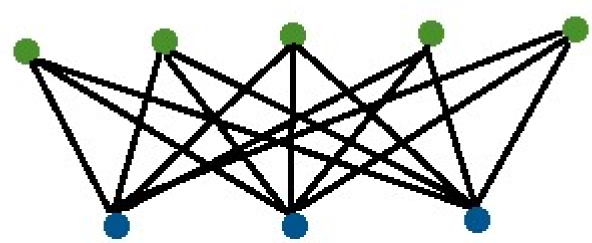
Remark

A simple graph with n vertices can have at most $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. The complete graph K_n has the maximum number of edges among all simple graphs with n vertices.

Complete Bipartite Graph

A graph $G = (V(G), E(G))$ is said to be Complete Bipartite if and only if there exists a partition $V(G) = A \cup B$ and $A \cap B = \emptyset$ so that all edges share a vertex from both set A and B and all possible edges that join vertices from set A to set B are drawn.

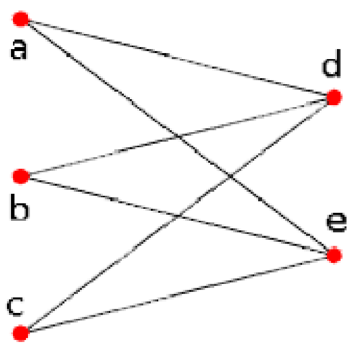
We denote a complete bipartite graph as $K_{r,s}$ where r refers to the number of vertices in subset A and s refers to the number of vertices in subset B.
For Example
Below is an example of the complete bipartite graph $K_{5,3}$:



Number of Vertices, Edges, and Degrees in Complete Bipartite Graphs

Since there are r vertices in set A, and s vertices in set B, and since $\mathrm{V(G)} = \mathrm{A} \cup \mathrm{B}$, then the number of vertices in $\mathrm{V(G)}$ is $|\mathrm{V(G)}\ | = \mathrm{r} + \mathrm{s}$.
Additionally, the number of edges in a complete bipartite graph is equal to $\mathrm{r} \cdot \mathrm{s}$ since r vertices in set A match up with s vertices in set B to form all possible edges for a complete bipartite graph.
Lastly, if the set A has r vertices and the set B has s vertices then all vertices in A have degree s, and all vertices in B have degree r. This should make sense since each vertex in set A connected to all s vertices in set B, and each vertex in set B connects to all r vertices in set A.
Every complete bipartite graph is not a complete graph.

Take for instance this graph.



This graph is clearly a bipartite graph. Moreover, it is a complete bipartite graph.

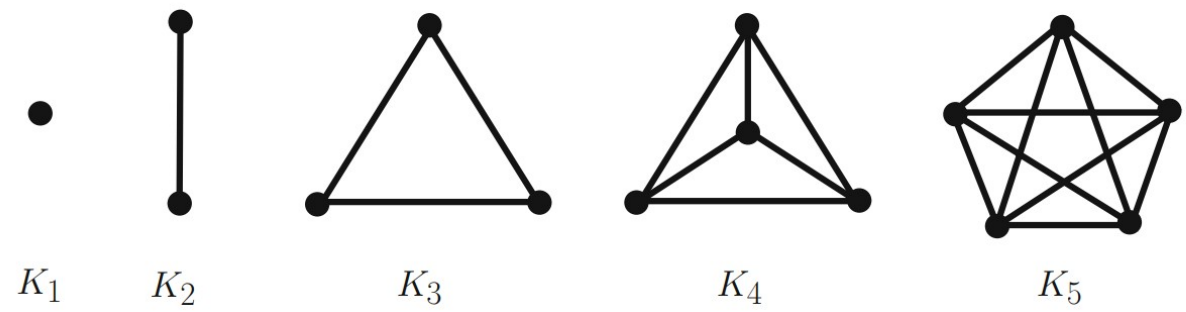
This graph is not a complete graph though as there is no edge between $\mathrm{a} - \mathrm{b}, \mathrm{b} - \mathrm{c}, \mathrm{a} - \mathrm{c}$, and $\mathrm{d} - \mathrm{e}$.

Remark

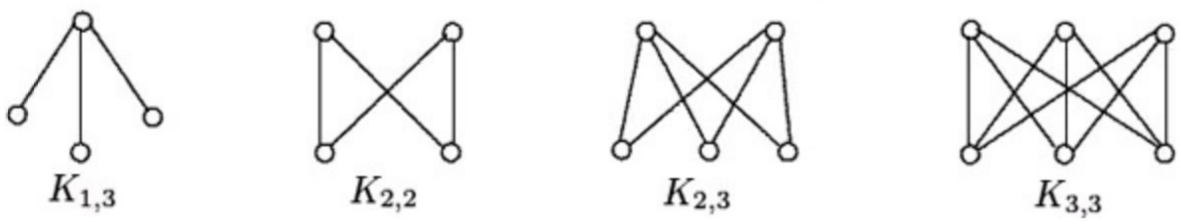
- The only non-trivial complete bipartite graph which is also a complete graph is K_1.
- $\mathrm{K}_{1,\mathrm{n}} = \mathrm{K}_{\mathrm{n},1}$ is a tree for all n, and no other complete bipartite graphs are trees.

Examples

Complete graphs



Complete bipartite graphs

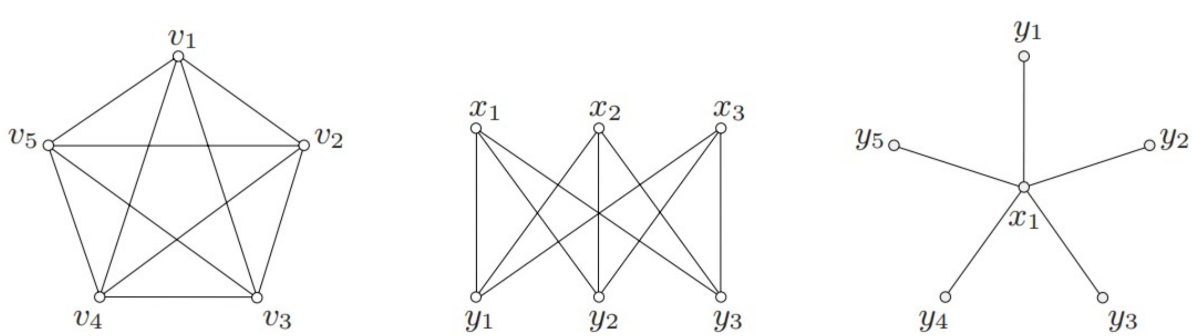


Remark

Complete bipartite graphs and complete graphs. The complete bipartite graph $K_{m,n}$ is a complete graph if and only if $m = n = 1$ or $\{m, n\} = \{1, 0\}$.

Example

In the figure is given below (a) is a complete graph (b) Complete Bipartite Graph and (c) is a bipartite Star Graph.



(a) (b) (c)

Clique

A clique of G is a complete subgraph of G: A clique of G is a maximal clique of G if it is not properly contained in another clique of G.

Complement of complete Graph

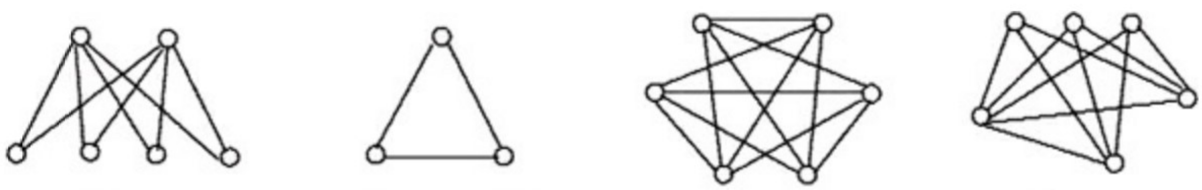
Compliment of the complete graph K_n then has n vertices and no edges; it is called the empty graph of order n.

Complete multipartite Graph

Bipartite graphs belong to a more general class of graphs. A graph G is a k-partite graph if V(G) can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ (once again called partite sets) such that if uv is an edge of G, then u and v belong to different partite sets. If, in addition, every two vertices in different partite sets are joined by an edge, then G is a complete k-partite graph. If $|V_i| = n_i$ for $1 \leq i \leq k$, then we denote this complete k-partite graph by $K_{n1,n2,\ldots,nk}$. The complete k-partite graphs are also referred to as complete multipartite graphs. If $n_i = 1$ for every $i\,(1 \leq i \leq k)$, then $K_{n1,n2,\ldots,nk}$ is the complete graph K_k . Complete 2-partite graphs are thus complete bipartite graphs.

Examples,

The following are some examples of Complete multipartite graphs



$K_{2,4}$ $\quad K_{1,1,1} = K_3$ $\quad K_{2,2,2}$ $\quad K_{1,2,3}$

Regular Graph

A k-regular graph G is one such that $\deg(v) = k$ for all $v \in G$.

Regular Bipartite Graph

A regular graph means that every vertex has the same degree this implies that (using the bipartite property) the number of vertices in $U = W$ again by it being bipartite and having vertices of equal degree implies that graph is Regular bipartite graph.

Result

A graph G is k-regular if $d(v) = k$ for all $v \in V$; a regular graph is one that is k-regular for some k. For instance, the complete graph on n vertices is $(n - 1)$ – regular, and the complete bipartite graph with k vertices in each part is k-regular.

Theorem

If G is a k-regular bipartite graph with $k > 0$ and the bipartition of G is X and Y, then the number of elements in X is equal to the number of elements in Y.

Proof

We observe $\sum_{v \in X} \deg(v) = k\,|X|$ and similarly, $\sum_{v \in Y} \deg(v) = k\,|Y|$. By the previous lemma, this means that $k\,|X| = k\,|Y| \Rightarrow |X| = |Y|$.