Mathematical Economics: Open and Closed Input-Output Models

Introduction:

The Input-Output Model is an interesting development in the field of modern economics. This model is used for national income accounts, planning and forecasting. There may be two types of input-output model- open and closed.

This module distinguishes the open input-output model from the closed model.

Objectives

The objectives of this module are:

1. *Explore* the mathematical structure of an input-output model
2. Differentiate between the open and closed input output model
3. Solve the input-output model using matrix algebra

Terminology

1. Open Input-Output Model: an input-output model with an "open sector" that consists of the demand for household consumption (H), Government consumption (G), Investment (I), Inventory (IN) and net exports (NE).
2. Closed Input-Output Model: an input-output model that includes only intermediate demand and no final demand
3. Final Demand: the demand for household consumption (H), Government consumption (G), Investment (I), Inventory (IN) and net exports (NE).
4. Input Coefficient: the ratio of the amount of output of one commodity used as intermediate input to produce one unit of output of some other commodity
5. Technology matrix: a matrix of the input coefficient
6. Leontief Inverse: the inverse of the technology matrix obtained by using matrix inverse
7. Homogenous equation system: an equation system in which RHS is equal to zero

18.1. Matrix Algebra and Input-Output Model

The Input-Output table shows the interdependence of various industries in an economy. Since the output of one industry is used as an input in other industries, the main concern is the level of output that should be produced by each industry so that the total demand for that product is fulfilled.

Suppose an economy has 3 industries and produces 3 commodities. Let Industry 1 produce commodity 1, industry 2 produce commodity 2 and industry 3 produce commodity 3 and assume that the industries are interdependent.

Let y_{ij} be the total quantity of i-th commodity needed as an input to produce the j-th commodity. Then, y_{11} is the total quantity of 1st commodity needed as an input in the production of the 1st commodity, y_{12} is the total quantity of 1st commodity needed as an input in the production of the 2nd commodity, y_{32} is the total quantity of 3rd commodity needed as an input in the production of the 2nd commodity and so on.

In tabular form the above information may be expressed as follows:

Output	**Industries**		
	Industry 1	**Industry 2**	**Industry 3**
Commodity 1	y_{11}	y_{12}	y_{13}
Commodity 2	y_{21}	y_{22}	y_{23}
Commodity 3	y_{31}	y_{32}	y_{33}

Table 18.1: Input-Output table for 3 industries and 3 commodities

Now,

$y_{11} + y_{12} + y_{13}$ gives the total quantity of the 1st commodity produced that is used as an input in all the three industries.

Similarly,

$y_{21} + y_{22} + y_{23}$ gives the total quantity of the 2nd commodity produced that is used as an input in all the three industries and

$y_{31} + y_{32} + y_{33}$ gives the total quantity of the 3rd commodity produced that is used as an input in all the three industries

Further,

$y_{11} + y_{21} + y_{31}$, $y_{12} + y_{22} + y_{32}$ and $y_{13} + y_{23} + y_{33}$ does not give meaningful result.

In real life, however, an economy has several other sectors. For example, the most important sector is the household sector. In that case, a part of the total quantity of a commodity produced by one industry also goes to the household sector. The demand made by the other sectors apart from the 3 industries is referred to as the final demand.

The next section explains in details an open input-output model in an economy and its solution using matrix algebra.

18.2. Open Input-Output Model and its solution using Matrix Algebra

If an input-output model consists of an "open sector" apart from the industries producing different commodities, then the model is termed as an open input-output model. The "open sector" may consist of the demand for household consumption (H), Government consumption (G), Investment (I), Inventory (IN) and net exports (NE). The demand from the open sector is generally termed as the Final Demand (D).

Therefore,

$$D = H + G + I + IN + NE$$

Now, the problem before the economy is to find the level of output of each industry such that the total demand for the output is fulfilled. This may be solved by using matrix algebra.

If Y_i is the total output of the i-th industry that would flow to other industries for intermediate demand and the open sector as final demand, then we may write

$$Y_i = y_{i1} + y_{i2} + y_{i3} + \dots\dots\dots\dots + y_{in} + D_i \qquad \text{(i= 1,2,3,....n)}$$

Here,

y_{i1} is the total quantity of output of the i-th sector used as input in the 1st sector,

y_{i2} is the total quantity of output of the i-th sector used as input in the 2nd sector and so on.

In general,

y_{ij} is the total quantity of output of i-th sector used as input in the j-th sector.

If there are 'n' industries in an economy, the flow of output of all the 'n' industries may be written as follows:

$$Y_1 = y_{11} + y_{12} + y_{13} + \dots\dots\dots\dots + y_{1n} + D_1$$

$$Y_2 = y_{21} + y_{22} + y_{23} + \dots\dots\dots\dots + y_{2n} + D_2$$

$$Y_3 = y_{31} + y_{32} + y_{33} + \dots\dots\dots\dots + y_{3n} + D_3$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Y_n = y_{n1} + y_{n2} + y_{n3} + \dots\dots\dots\dots + y_{nn} + D_n$$

A. Input-Coefficient:

Let us assume that the output is in terms of money.

If we divide y_{11} by Y_1, the ratio $\frac{y_{11}}{Y_1}$ will give the amount (in terms of money) of the 1st commodity used as intermediate input to produce one unit of output of the 1st commodity.

Similarly, $\frac{y_{21}}{Y_1}$ will give the amount of the 2nd commodity used as intermediate input to produce one unit of output of the 1st commodity and so on.

In general, $\frac{y_{ij}}{Y_j}$ will give the amount of output of the i-th commodity used as intermediate input to produce one unit of output of the j-th commodity.

Let $$\frac{y_{ij}}{Y_j} = b_{ij} \quad \text{---------------------- (18.1)}$$

Then, $$y_{ij} = b_{ij}Y_j \quad \text{---------------------- (18.2)}$$

b_{ij} is known as the input-coefficient or technical co-efficient.

B. Solution using Matrix Algebra:

Using equation (1) in the system of simultaneous equations, we get

$$Y_1 = b_{11}Y_1 + b_{12}Y_2 + b_{13}Y_3 + \dots\dots\dots\dots + b_{1n}Y_n + D_1$$

$$Y_2 = b_{21}Y_1 + b_{22}Y_2 + b_{23}Y_3 + \dots\dots\dots\dots + b_{2n}Y_n + D_2$$

$$Y_3 = b_{31}Y_1 + b_{32}Y_2 + b_{33}Y_3 + \dots\dots\dots\dots + b_{3n}Y_n + D_3$$

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$$Y_n = b_{n1}Y_1 + b_{n2}Y_2 + b_{n3}Y_3 + \dots\dots\dots\dots + b_{nn}Y_n + D_n$$

In order to solve the above simultaneous equation, we may use matrix algebra.

Expressing the simultaneous equations in matrix form, we may write as follow:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ . \\ . \\ . \\ Y_n \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \dots \dots .. b_{1n} \\ b_{21} & b_{22} & b_{23} \dots \dots \dots b_{2n} \\ b_{31} & b_{32} & b_{33} \dots \dots \dots b_{3n} \\ \dots & \dots & \dots \dots \dots \dots \dots . \\ \dots & \dots & \dots \dots \dots \dots \dots . \\ \dots & \dots & \dots \dots \dots \dots \dots . \\ b_{n1} & b_{n2} & b_{n3} \dots \dots \dots b_{nn} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ . \\ . \\ . \\ Y_n \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ . \\ . \\ . \\ D_n \end{bmatrix}$$

Let

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ . \\ . \\ . \\ Y_n \end{bmatrix} \qquad b = \begin{bmatrix} b_{11} & b_{12} & b_{13} \dots \dots .. b_{1n} \\ b_{21} & b_{22} & b_{23} \dots \dots \dots b_{2n} \\ b_{31} & b_{32} & b_{33} \dots \dots \dots b_{3n} \\ \dots & \dots & \dots \dots \dots \dots \dots . \\ \dots & \dots & \dots \dots \dots \dots \dots . \\ \dots & \dots & \dots \dots \dots \dots \dots . \\ b_{n1} & b_{n2} & b_{n3} \dots \dots \dots b_{nn} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ . \\ . \\ . \\ D_n \end{bmatrix}$$

Y is a column vector of order $n \times 1$ and the elements are the total output of each industry that needs to be produced to fulfill the total demand.

b is a matrix of order $n \times n$ and is known as the coefficient matrix. By definition of equation (1), the input coefficient matrix shows the output of each industry used as input to produce one unit of every other industry.

D is a column vector of order $n \times 1$and the elements are the final demand of each industry.

Therefore we can write the problem of input-output in matrix form as;

$$Y = bY + D$$

Or, $(I - b)Y = D$ here I is an Identity matrix of order $n \times n$

Or, $Y = (I - b)^{-1}D$ -------------- (18.3)

C. Technology Matrix and Leontief Inverse matrix:

In equation (3)

$$I - b = \begin{bmatrix} 1 - b_{11} & -b_{12} & -b_{13} \dots \dots .. -b_{1n} \\ -b_{21} & 1 - b_{22} & -b_{23} \dots \dots \dots -b_{2n} \\ -b_{31} & -b_{32} & 1 - b_{33} \dots \dots \dots - b_{3n} \\ & \dots \dots \dots \dots \dots \dots \dots . & \\ & \dots \dots \dots \dots \dots \dots \dots . & \\ & \dots \dots \dots \dots \dots \dots \dots . & \\ -b_{n1} & - b_{n2} & -b_{n3} \dots \dots \dots 1 - b_{nn} \end{bmatrix}$$ is known as the **Technology Matrix**

The inverse of the technology matrix or $(I - b)^{-1}$ obtained by using the Inverse formula is known as **Leontief Inverse Matrix.**

By using equation (3) and the method of matrix inversion, we may calculate the output of each commodity that needs to be produced so that the total demand is fulfilled.

Note:

1. If an industry does not use its output as an input in its own industry, or in any other industry, the input coefficient will be zero.
2. The row-sum $b_{11} + b_{12} + b_{13} \dots \dots . + b_{1n}$ will give the amount of the 1st commodity required as input to produce one unit each of commodity 1, 2, 3 ….. n respectively.

   But the inputs required by all the industries to produce one unit of output of each commodity do not carry any economic meaning. Hence the row-sum of the input coefficient matrix does not carry any economic meaning.

   However, if each of the coefficients are multiplied by the respective total outputs, that is,

   $b_{11}Y_1 + b_{12}Y_2 + b_{13}Y_3 \dots \dots . + b_{1n}Y_n$ will give the total amount of Y_1 needed as input for all the n industries.

18.3. Closed Input-Output Model

The example of 3 industries discussed in section 6.2.1 is an example of a closed input-output model. In such a model, the column of final demand that was included in the open input-output model as discussed in section 6.2.2 is not considered. Therefore there exists only intermediate demand. That is, all the output of all the industries will be produced only to be used by all the other industries in the economy.

Mathematically, this will bring about a change in the system of simultaneous equations. In the presence of final demand, the system of simultaneous equations was written as

$$Y_1 = y_{11} + y_{12} + y_{13} + \dots \dots \dots \dots + y_{1n} + D_1$$

$$Y_2 = y_{21} + y_{22} + y_{23} + \dots \dots \dots \dots + y_{2n} + D_2$$

$$Y_3 = y_{31} + y_{32} + y_{33} + \dots \dots \dots \dots + y_{3n} + D_3$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Y_n = y_{n1} + y_{n2} + y_{n3} + \dots \dots \dots \dots + y_{nn} + D_n$$

When the final demands disappear, the system becomes

$$Y_1 = y_{11} + y_{12} + y_{13} + \dots\dots\dots\dots + y_{1n}$$

$$Y_2 = y_{21} + y_{22} + y_{23} + \dots\dots\dots\dots + y_{2n}$$

$$Y_3 = y_{31} + y_{32} + y_{33} + \dots\dots\dots\dots + y_{3n}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Y_n = y_{n1} + y_{n2} + y_{n3} + \dots\dots\dots\dots + y_{nn}$$

Or,

$$Y_1 = b_{11}Y_1 + b_{12}Y_2 + b_{13}Y_3 + \dots\dots\dots\dots + b_{1n}Y_n$$

$$Y_2 = b_{21}Y_1 + b_{22}Y_2 + b_{23}Y_3 + \dots\dots\dots\dots + b_{2n}Y_n$$

$$Y_3 = b_{31}Y_1 + b_{32}Y_2 + b_{33}Y_3 + \dots\dots\dots\dots + b_{3n}Y_n$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Y_n = b_{n1}Y_1 + b_{n2}Y_2 + b_{n3}Y_3 + \dots\dots\dots\dots + b_{nn}Y_n$$

In matrix form,

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ . \\ . \\ . \\ Y_n \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \dots\dots.. b_{1n} \\ b_{21} & b_{22} & b_{23} \dots\dots\dots b_{2n} \\ b_{31} & b_{32} & b_{33} \dots\dots\dots b_{3n} \\ \dots & \dots & \dots\dots\dots\dots\dots\dots \\ \dots & \dots & \dots\dots\dots\dots\dots\dots \\ \dots & \dots & \dots\dots\dots\dots\dots\dots \\ b_{n1} & b_{n2} & b_{n3} \dots\dots\dots b_{nn} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ . \\ . \\ . \\ Y_n \end{bmatrix}$$

Or, $Y = bY$

Or, $Y - bY = 0$ (0 is a zero-column vector of order $n \times 1$)

Or, $(I - b)Y = 0$

Or, $Y = (I - b)^{-1}0$

Or, $Y = 0$ --------------------- (18.4)

Equation (18.4) is a homogenous equation system.

It must be understood that $|I - b|$ may not be equal to zero. Therefore, it may be said that a solution exists but is trivial which means that there is no unique solution to the problem. In other words, no unique output mix exists for the industries.

In fact, a homogenous equation system may have infinite number of solutions.

18.4. Note on homogeneous equation system:

"Homogenous" means same or alike. The property of homogeneity in mathematics means that if all the variables are multiplied by the same number, the equation system will remain valid.

A function $f\ (x, y)$ will be homogenous if $f\ (\lambda x, \lambda y) = \lambda^{\alpha} f(x, y)$.

α is a real number and is known as the degree of homogeneity.

In the case of the closed input-output model,

$$(I - b)Y = 0 \text{ ------------------- } (18.5)$$

is a special case where the right hand side is equal to a zero vector. This special case is referred to as a 'homogeneous equation system' as the right-hand side is not a scalar but a vector whose elements are zero.

Equation 18.5 may be written as

$$Y = \ (I - b)^{-1} 0$$

Note that the inverse may not be equal to zero.

Example 1: When determinant is not equal to zero

$$2y_1 + y_2 = 0$$

$$4y_1 + 3y_2 = 0$$

Using matrix method, we may write

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Or, $$Y = \ b^{-1} D$$

We know

$$b^{-1} = \frac{Adj\ b}{|b|}$$

Now, $Adj\ b = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ and $|b| = 2$

Or,

$$b^{-1} = \frac{\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}}{2} = \begin{bmatrix} 1 & 1/2 \\ 2 & 3/2 \end{bmatrix}$$

Clearly, $y_1 = y_2 = 0$

Example 2: When determinant is equal to zero

$$2y_1 + y_2 = 0$$

$$4y_1 + 2y_2 = 0$$

Using matrix method, we may write

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Or, $$Y = b^{-1}D$$

We know

$$b^{-1} = \frac{Adj\ b}{|b|}$$

Now, $Adj\ b = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ and $|b| = 0$

Since the determinant is zero, by definition of singularity of a matrix, the rows in the coefficient matrix are linearly dependent and one of the equations is redundant.

Ignoring the second equation, we get

$$y_2 = -2y_1 \text{ and } y_1 = -\frac{y_2}{2}$$

Clearly it can be seen that there will be infinite solutions.

