Mathematical Economics: Optimization-Maxima and Minima

21.1 Introduction:

Human beings have a tendency to thrive for excellence or give the best performance in different aspects of life. "Satisfaction" is one thing and "optimum satisfaction" is another thing. "Satisfaction" is like saying "I am happy"; "Optimum satisfaction" is like saying "I am on top of the world".

This module will explain the technique of arriving at the best decision.

Objectives

The objectives of this module are:

1. *Define the* concept of a maxima and a minima.
2. *Explain the technique of optimizing* without constraint and with constraint

Terminology

1. Optimization: a technique of obtaining the optimum
2. Optimum: best
3. Equilibrium: a state of balance; not necessarily the best
4. Maximum: highest or greatest
5. Minimum: lowest or smallest
6. Inflection Point: a point where a curve changes shape

21.2. Meaning of Optimization

Optimization is a technique or method of obtaining the optimum. The word "optimum" comes from the Latin word "*optimus*" which means "best". Newton and Gauss proposed iterative methods for moving towards an optimum. It was Pierre De Fermat and Joseph Louise Lagrange who founded calculus-based formulae for identifying optima.

Image 21.1 Pierre De Fermat
[Source: https://en.wikipedia.org/wiki/File:Pierre_de_Fermat.jpg]

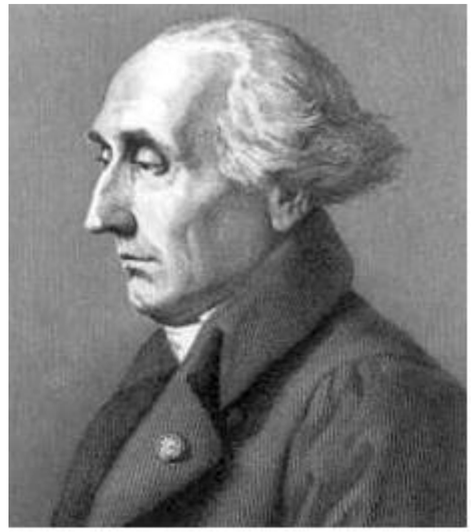


Image 21.2 Joseph Louise Lagrange
[Source: https://en.wikipedia.org/wiki/File:Lagrange_portrait.jpg]

The word "optimization" is known to be first used in 1857. Optimization is synonymously used with mathematical programming. The techniques and methods of optimization are used in diverse fields such as physics, biology, engineering, economics and business. The invention of computers has led to the development of optimization techniques. Apart from computer science, optimization techniques are also used to solve quantitative problems in operations research, game theory, numerical analysis and control theory.

21.3. Some examples the application of optimization techniques: Few examples where the techniques of optimization are used are given below:

1. **Design Optimization** is the process of finding the best design out of alternative designs. In designing an aircraft, designers normally model an initial design with consideration of all the constraints on their design.
2. **Filter design** in electrical engineering is the process of designing a signal processing filter (process of removing unwanted features in a signal) that satisfies a set of requirements. It is an optimization problem where each requirement contributes to an error function that should be minimized.

3. **Operations management** uses the technique of optimization to manage strategic and day-to-day production of goods and services. For example, optimization technique is used to minimize the waiting time for customers.
4. **Economic Optimization** such as Utility Maximization, Cost Minimization, Revenue maximization, etc use optimization techniques in order to satisfy the economic agents. International Trade theories use optimization techniques to explain the patterns of trade between nations. Maximizing the returns and minimizing the risks in allocation of financial assets such as stocks, bonds and cash is an example of where multi-objective optimization problem that involve multiple objective functions.

21.4. Difference between Equilibrium and Optimization

Equilibrium is a state of rest where economic forces are in balance. In the market equilibrium and national income equilibrium, there is no particular objective of attaining the best solution. The market is said to be balanced when demand equals the supply. The consumers and producers arrive at a market equilibrium that is based on the adjustments between the demand and supply. In this case, the consumers and producers do not have any objective of attaining a particular price. Such type of equilibrium is referred as “non-goal equilibrium”.

Optimization is a special type of equilibrium in which there is an objective to be fulfilled. It may be referred as “goal equilibrium”.

For example, utility maximization requires finding the quantity of goods that should be consumed, given a budget constraint that would maximize the satisfaction of the consumer. In this case, it is not a case of a balance between consumption and budget, but the objective is to find that particular bundle of the goods that should be consumed so that the consumer attains maximum utility.

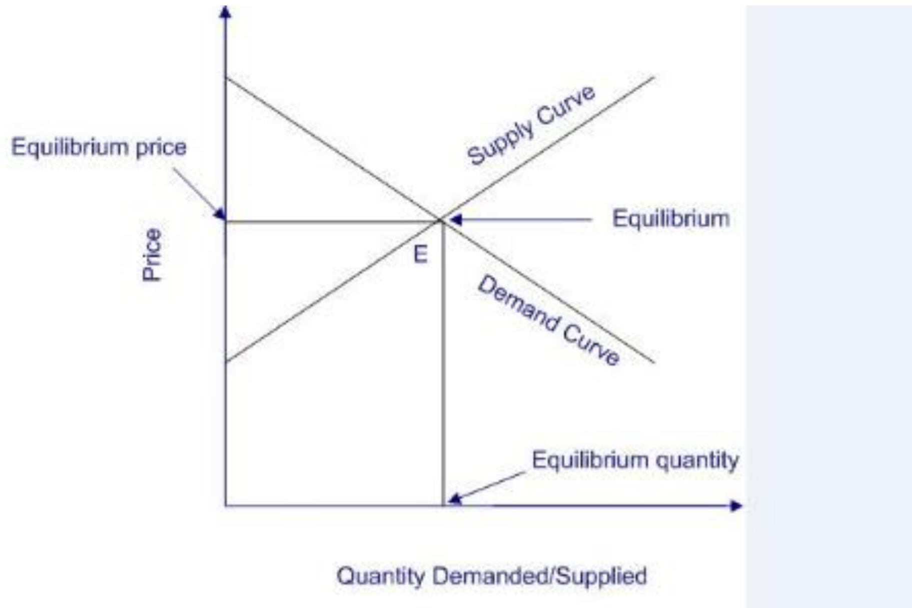


Fig. 21.1: Market Equilibrium (Non-goal equilibrium)

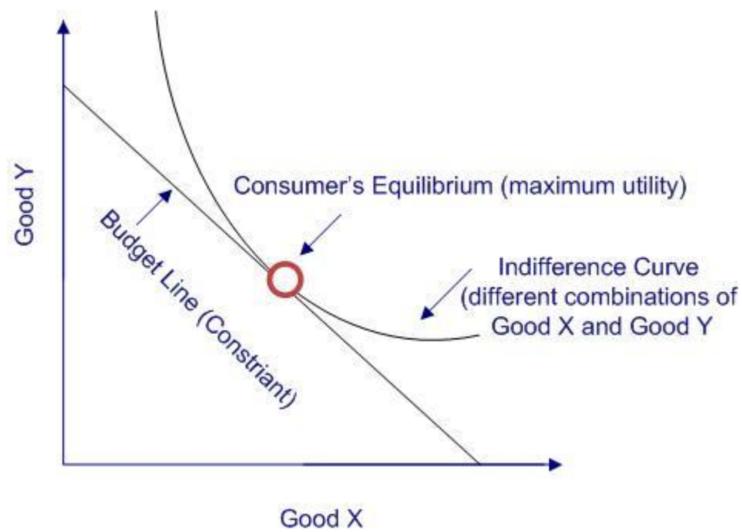


Fig. 21.2: Consumer's Equilibrium (Goal equilibrium)

In Fig. 21.1, there is a demand curve and supply curve and both the forces of demand and supply come together to attain a balance or an equilibrium. There is no objective of such a situation. This is a case of non-goal equilibrium.

In Fig.21.2, the objective is to find the combination of Good X and Good Y such that the utility of the consumer is maximized, given the budget of the consumer as the constraint or restriction. This is a case of goal equilibrium or optimization.

21.5. Concept of Maxima and Minima

Fermat developed a technique called "Adequality" to calculate the maxima and minima of a function. 'Maxima' is the plural form or 'maximum' and 'minima' is the plural form of 'minimum'. In mathematics, the technical word that is used to represent maximum or minimum values is 'extrema' which means 'an extreme value' within a given range or domain of a function.

A function $y = f(x)$ may have a global maximum or a global minimum and a local maximum or a local minimum value. The maximum value or minimum value for the entire range of the function is called the global extremum. The maximum or minimum value in the immediate neighborhood of that point only is the local maximum or local minimum. The concept of global and local maxima and minima is shown in Image 5.

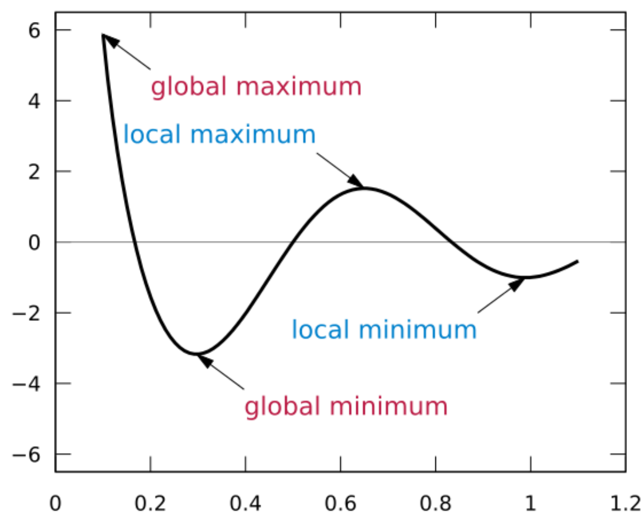


Image 21.7: Global and Local Maxima and Minima

[Source: https://en.wikipedia.org/wiki/File:Extrema_example_original.svg **]**

The maxima and minima can be expressed in terms of derivative. By using the first derivative and second derivative of a function, we can obtain the necessary and sufficient conditions for maxima or minima.

Consider the following graph of the function with one explanatory variable $y = f(x)$

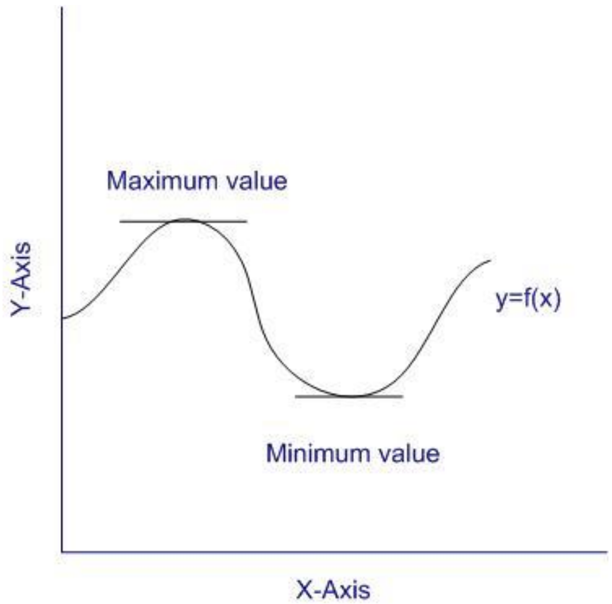


Fig. 21.3: Conditions for maxima and minima

Following are the observations from Fig. 21.3:

1. The tangent at the maximum and the minimum is horizontal. That is the slope of the tangent at the maximum and minimum is zero
2. Before attaining a maximum point, the curve slopes upward at a decreasing rate and after attaining a maximum point, the curve slopes downward at a decreasing rate.
3. Before attaining a minimum point, the curve slopes downward at an increasing rate and after attaining a minimum point, the curve slopes upward at an increasing rate.

We learned that the slope of a function is the ratio of the change in the dependent variable to the change in the independent variable. The sign of the first derivative $\frac{dy}{dx}$ gives the value of the function $y = f(x)$. The sign of the second derivative $\frac{d}{dx}(\frac{dy}{dx})$ or $\frac{d^2y}{dx^2}$ gives the slope of the curve.

The following table gives the possible relationships between the first derivative and the second derivative (for a function with one explanatory variable) and their interpretations:

Table 21.1: Conditions for Maxima and Minima (One Explanatory variable)

S.No.	Sign of first derivative	Sign of second derivative	Interpretation
1	$\frac{dy}{dx} > 0$	$\frac{d^2y}{dx^2} < 0$	Value of the function tends to increase at a decreasing rate or it is upward concave
2	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} < 0$	Value of the function has reached a maximum point
3	$\frac{dy}{dx} < 0$	$\frac{d^2y}{dx^2} < 0$	Value of the function tends to is decrease at a decreasing rate or it is downward concave
4	$\frac{dy}{dx} < 0$	$\frac{d^2y}{dx^2} > 0$	Value of the function tends to decrease at an increasing rate or it is downward convex
5	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} > 0$	Value of the function has reached a minimum point
6	$\frac{dy}{dx} > 0$	$\frac{d^2y}{dx^2} > 0$	Value of the function tends to increase at an increasing rate or it is upward convex
7.	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} = 0$	Neither maximum or minimum (Stationary inflection point)
8.	$\frac{dy}{dx} \neq 0$	$\frac{d^2y}{dx^2} = 0$	Neither maximum or minimum (Non-stationary inflection point)

Therefore, for a function $y = f(x)$,

a) Necessary condition for a maximum or a minimum is $\frac{dy}{dx} = 0$

b) Sufficient condition for a maximum is $\frac{d^2y}{dx^2} < 0$ and for a minimum is $\frac{d^2y}{dx^2} > 0$

The curvature of the graph using the second derivative is shown in Fig. 21.4 and Fig. 21.5

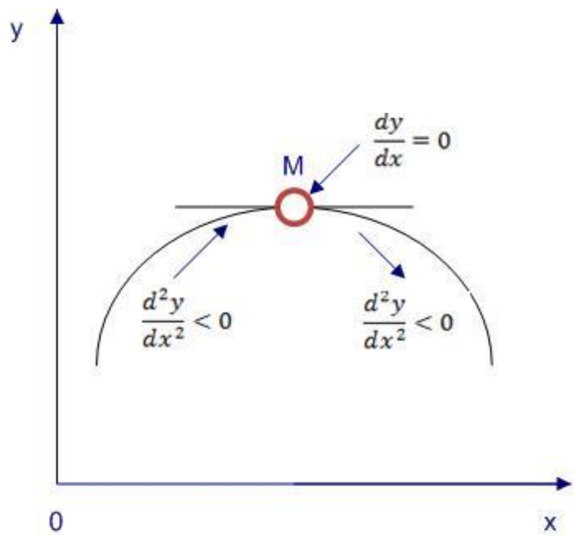


Fig. 21.4: Concave Curve

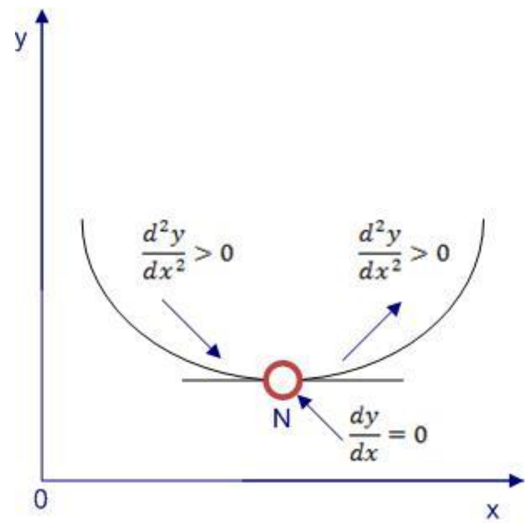


Fig. 21.5: Convex Curve

From Fig. 21.4 and Fig.21.5 it is seen that the second derivative does not change sign before and after the graph reaches an extreme point. Fig. 21.4 has a maximum and shows a concave curve and Fig.21. 5 has a minimum and shows a convex curve.

Note:

In case where a function has a maxima or a minima, the curvature does not change. For example, in Table 21.1, for maxima, the second derivative before and after $\frac{dy}{dx} = 0$, is $\frac{d^2y}{dx^2} < 0$ and it results in a concave curve for the function. The case is reversed for a convex curve.

21.6. Inflection Point:

In context with mathematics, inflection means an act of curving or bending. Since the second derivative gives an idea about the curvature, in terms of the second derivative, the inflection point may be characterized as a point where the curvature undergoes a change. That is, a concave curve turns to a convex curve and a convex curve turns to a concave curve.

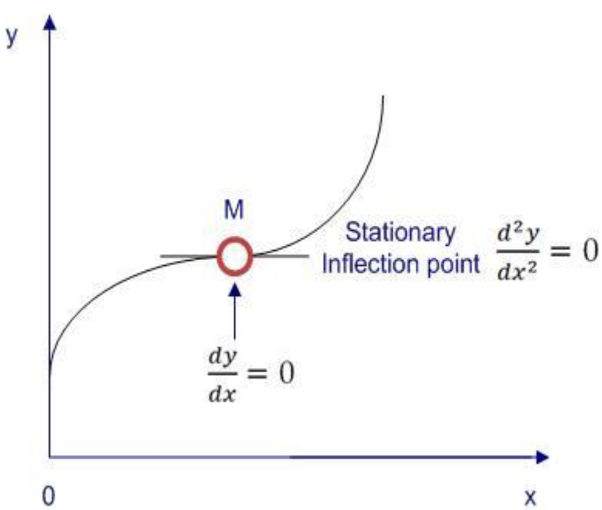
By using the concept of first and second derivative, we may come across two different situations:

1. $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$

If the first order derivative is equal to zero, this means that the slope of the function is a horizontal straight line (since the change in y is zero). This means that function has attained a stationary point. This is a case of ***stationary inflection point***. Since the second derivative is equal to zero at the point where $\frac{dy}{dx} = 0$, we do not have a maximum or a minimum and if $\frac{d^2y}{dx^2} < 0$ changes to $\frac{d^2y}{dx^2} > 0$, the graph changes its curvature from concave to convex. (See Fig. 21.6)

2. $\frac{dy}{dx} \neq 0$ and $\frac{d^2y}{dx^2} = 0$

If the first derivative is not equal to zero, this means that the change in y is not zero and the function does not attain a stationary point. It may be greater than zero or less than zero. This is a case of ***non-stationary inflection point***. If $\frac{dy}{dx} > 0$, it means that the slope of the function increases up to a certain point and then decreases, that is $\frac{dy}{dx} > 0$. Since the second derivative is equal to zero, we again do not have a maximum or a minimum and if $\frac{d^2y}{dx^2} > 0$ changes to $\frac{d^2y}{dx^2} < 0$ the graph changes its curvature from convex to concave. (See Fig. 21.7)



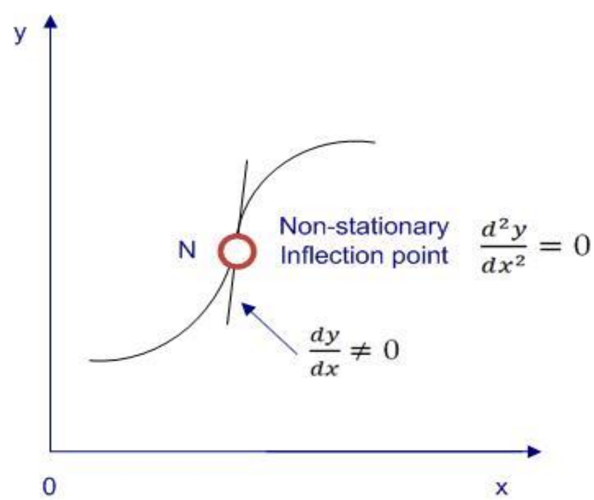


Fig. 21.6: Stationary Inflection Point **Fig. 21.7: Non-stationary Inflection Point**

21.7. Optimization without any constraint

This section aims at finding the optimum values of a function (using numerical problems) in two situations. These situations are discussed below.

A. Optimization with one explanatory variable

Let us consider a function with one explanatory variable $y = f(x)$ such that

$$y = 5x^2 - 40x + 20 \text{ ----------- } (21.1)$$

To determine the extreme value (maximum or minimum)

1. First order condition requires $\frac{dy}{dx} = 0$

   Now,

$$\frac{dy}{dx} = \frac{d}{dx}(5x^2 - 40x + 20)$$

$$or, \quad \frac{dy}{dx} = 10x - 40 \ \ (using\ basic\ rules\ of\ differentiation)$$

   Therefore, $\frac{dy}{dx} = 0$ => $10x - 40 = 0$ => $x = 4$

2. To check if the function has a maximum or a minimum, we find the second derivative.

   Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(10x - 40)\ or, \qquad \frac{d^2y}{dx^2} = 10 > 0$$

   Therefore, $x = 4$ minimizes the function $y = 5x^2 - 40x + 20$.

An example from Economic Theory:

Problem of maximization:

The main objective of producers is to maximize their profit. The total cost (TC) and total revenue (TR) depends on the output (Q) produced and sold. According to the Theory of Production, a producer is in equilibrium under two conditions:

1) Marginal Revenue (MR) is equal to Marginal Cost (MC)
2) Marginal Cost curve cuts Marginal Revenue curve from below

Also, profit (π) is defined as the difference between total revenue and total cost.

The economic problem is to obtain the amount of output that will maximize the profit. There is no constraint.

The solution:

Translating the producer's theory in mathematical form we have the following functions:

a) $TC = f(Q), MC = \frac{d(TC)}{dQ}$

b) $TR = f(Q), MR = \frac{d(TR)}{dQ}$

c) $\pi = TR - TC \text{ or } \pi = f(Q)$

d) $Slope\ of\ MC = \frac{d^2(TC)}{dQ^2}$

e) $Slope\ of\ MR = \frac{d^2(TR)}{dQ^2}$

To maximize the profit,

i) First order condition requires

$$\frac{d\pi}{dQ} = 0 \text{ or } \frac{d}{dQ}(TR - TC) = 0 \text{ or } \frac{d(TR)}{dQ} = \frac{d(TC)}{dQ}$$

This is the first order condition for producer's equilibrium, that is, MR =MC

ii) Second order condition for maximization requires

$$\frac{d^2\pi}{dQ^2} < 0 \text{ or } \frac{d}{dQ}\left(\frac{d\pi}{dQ}\right) < 0 \text{ or } \frac{d^2(TR)}{dQ^2} < \frac{d^2(TC)}{dQ^2}$$

This is the second order condition for producer's equilibrium, that is, Slope of MR < Slope of MC

B. Optimization with more than one explanatory variable

Let us consider a function with two explanatory variables $z = f(x, y)$. To determine the extreme values for a function with more than one explanatory variable, Hessian Determinant is used. The necessary and sufficient conditions for maxima and minima with more than one explanatory variable are as follows:

a) Necessary condition for maxima or minima requires the two partial derivatives must simultaneously be equal to zero. That is,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$$

b) Sufficient condition for maxima requires $|H_1| < 0, |H_2| > 0, \dots\dots\dots (-1)^m|H_m| > 0$

for minima requires $|H_1| > 0, |H_2| > 0, \dots\dots\dots |H_m| > 0$

$$'m' \text{ is the number of explanatory variables in the function}$$

Example:

Let us consider a function with two explanatory variables such that

$$z = 3x^2 - 12x + 4xy + 2y^2 - 8y + 100 \text{ ----------------- } (21.2)$$

To determine the extreme value (maximum or minimum)

(1) First order condition requires

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \text{ -------------------------- } (21.3)$$

Now,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(3x^2 - 12x + 4xy + 2y^2 - 8y + 100\ )$$

$$or, \quad \frac{\partial z}{\partial x} = 6x - 12 + 4y - - - - - - - - (21.4)$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(3x^2 - 12x + 4xy + 2y^2 - 8y + 100\ )$$

$$or, \quad \frac{\partial z}{\partial y} = 4x + 4y - 8 - - - - - - - - - (21.5)$$

Using equation (21.4) and (21.5)

$$6x - 12 + 4y = 0 \text{ -------------------------- } (21.6)$$

$$and \quad 4x + 4y - 8 = 0 \quad \text{ ----------------------- } (21.7)$$

Solving equation (21.6) and (21.7) gives

$$x = 2 \ and \ y = 0$$

(2) To check if the function has a maximum or a minimum, we test the second order condition.

We know that for maxima, $|H_1| < 0, |H_2| > 0, \dots \dots \dots (-1)^m |H_m| > 0$

and for minima $|H_1| > 0, |H_2| > 0, \dots \dots \dots |H_m| > 0$

Now, from concept of Hessian determinant,

$$|H_1| = \frac{\partial^2 z}{\partial x^2}$$

$$or, |H_1| = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$or, |H_1| = \frac{\partial}{\partial x}(6x - 12 + 4y)$$

$$or, |H_1| = 6 > 0$$

And

$$|H_2| = \begin{vmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{vmatrix}$$

Using equation (21.4) and (21.5) we get,

$$\frac{\partial^2 z}{\partial x^2} = 6, \quad \frac{\partial^2 z}{\partial x \partial y} = 4, \quad \frac{\partial^2 z}{\partial y \partial x} = 4, \quad \frac{\partial^2 z}{\partial y^2} = 4$$

$$\therefore \ |H_2| = \begin{vmatrix} 6 & 4 \\ 4 & 4 \end{vmatrix} = 24 - 16 = 8 > 0$$

Hence, at $x = 2 \ and \ y = 0$, the function is minimized.

Examples from Economic Theory:

The concept of optimization with more than one explanatory variable may be used to solve many economic problems. The most common economic problems are:

a) Profit maximization for a discriminating monopolist
b) Profit maximization of a multi-product firm
c) Profit maximization of a multi-plant firm
d) Cost Minimization and Optimum input combination

(Numerical problems related to the above examples are explained in details in Module 22)

21.8. Optimization with one equality constraint:

In the previous section, we learned how to find the extreme values for functions without any constraint. In this section we will learn how to find the extreme values for functions with one equality constraint. Here, the concept of bordered Hessian determinant is used.

Let us consider a function with two explanatory variables $z = f(x, y)$ subject to an equality constraint $h(x, y) = c$

To find the optimum solution, we construct the Lagrange function

$$L = f(x,y) + \lambda[c - h(x,y)]$$
$$where\ \ \lambda\ is\ known\ as\ the\ Lagrange\ multiplier$$

To determine the optimum values,

1) First order condition requires

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$$

2) Second order condition

   for maximization requires $|\overline{H_2}| > 0$, $|\overline{H_3}| < 0 \dots \dots \dots (-1)^m|\overline{H_m}| > 0$

   for minimization requires $|\overline{H_2}| < 0\ or\ |\overline{H_3}| < 0 \dots \dots \dots |\overline{H_m}| < 0$

Example:

Let us consider a function $z = 2x + 2xy + y$ subject to a constraint $3x + 2y = 12$

To find the optimum solution, we construct the Lagrange function

$$L = 2x + 2xy + y + \lambda(12 - 3x - 2y)$$
$$where\ \ \lambda\ is\ known\ as\ the\ Lagrange\ multiplier$$

To determine the optimum values,

1) First order condition requires

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0 - - - - - - - - (21.8)$$

   Therefore,

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}[2x + 2xy + y + \lambda(12 - 3x - 2y)]$$

$$or, \quad \frac{\partial L}{\partial x} = 2 + 2y - 3\lambda - - - - - - - - - - (21.9)$$

$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}[2x + 2xy + y + \lambda(12 - 3x - 2y)]$$

$$or, \quad \frac{\partial L}{\partial y} = 2x + 1 - 2\lambda - - - - - - - - - - (21.10)$$

$$and \quad \frac{\partial L}{\partial \lambda} = \frac{\partial}{\partial \lambda}[2x + 2xy + y + \lambda(12 - 3x - 2y)]$$

$$or, \quad \frac{\partial L}{\partial \lambda} = 12 - 3x - 2y - - - - - - - - - - (21.11)$$

Using equation (21.8) and solving equation (21.9), (21.10) and (21.11)

$$2 + 2y - 3\lambda = 0 - - - - - (21.12)$$

$$2x + 1 - 2\lambda = 0 - - - - - (21.13)$$

$$12 - 3x - 2y = 0 - - - - - - (21.14)$$

Solving equation (21.12) and (21.13), we get,

$$\frac{2}{3} + \frac{2y}{3} = x + \frac{1}{2}$$

$$or, \quad \frac{(2 + 2y)}{3} = \frac{(2x + 1)}{2}$$

$$or, \quad 6x - 4y = 1 - - - - - - - - - (21.15)$$

Solving equation (21.14) and (21.15), we get,

$$3x + 2y = 12$$

$$6x - 4y = 1$$

Or,

$$x = \frac{25}{2} \; and \; y = \frac{23}{8}$$

2) To check if the function has a maximum or a minimum, we test the second order condition

We know that for maxima $|\overline{H_2}| > 0$, $|\overline{H_3}| < 0 \dots \dots \dots (-1)^m |\overline{H_m}| > 0$

and for minima $|\overline{H_2}| < 0 \; or \; |\overline{H_3}| < 0 \dots \dots \dots |\overline{H_m}| < 0$

Now,

$$|\overline{H_2}| = \begin{vmatrix} 0 & h_1 & h_2 \\ h_1 & L_{11} & L_{12} \\ h_2 & L_{21} & L_{22} \end{vmatrix}$$

Where,

$$h_1 = \frac{\partial h}{\partial x}, \; h_2 = \frac{\partial h}{\partial y}, \; L_{11} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial x}\right), \; L_{12} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y}\right), \; L_{21} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial x}\right), L_{22} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial y}\right)$$

Using equation (9), (10) and (11)

$$|\overline{H_2}| = \begin{vmatrix} 0 & 3 & 2 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix} = 0(0 - 4) - 3(0 - 4) + 2(6 - 0) = 24 > 0$$

Therefore, at $x = \frac{25}{2} \; and \; y = \frac{23}{8}$, the function is maximized.

