Solved Problems in Linear Programming

26.1 Maximization Problem

Tata Steel is an Indian multi-national steel-making company. It produces a large number of products. Consider two products x_1 and x_2. The production of x_1 takes 6 hours for rolling, 3 hours for cutting and one hour for coating. The production of x_2 takes 2 hours for rolling, 5 hours for cutting and 4 hours for coating. The plant has 36 hours of rolling time available, 30 hours of cutting time and 20 hours of coating time. The rates of profit per unit of x_1 and x_2 are ₹ 10 thousand and ₹ 8 thousand respectively.

a) Formulate a mathematical model for the given information?

b) According to your analysis, what level of output should Tata Steel produce so as to maximize profit? Use the graphical method to explain your analysis.

c) What is the maximum profit?

Solution:

a) Let us first organize the data in a table.

Table 26.1: Details of processing time and rate of profit

Stages of production	Time required for each phase of processing one unit (in machine hours)		Maximum time available (in machine hours)
	x_1	x_2	
Rolling	6	2	36
Cutting	3	5	30
Coating	1	4	20
Profit per unit (in ’000 ₹)	10	8	

Clearly, the objective of the plant is to maximize the profit given three constraints. Using simple mathematical function between the variables, the objective function and the constraints may be written in mathematical language as

Maximize

$$\pi = 10x_1 + 8x_2, \qquad \pi \text{ is the profit}$$

subject to

$$6x_1 + 2x_2 \leq 36$$

$$3x_1 + 5x_2 \leq 30$$

$$x_1 + 4x_2 \leq 20$$

The model just constructed is a linear programming problem with inequality constraints. The graphical analysis for solving the problem requires us to draw the graphs of the constraints and find the feasible region and then arrive at the solution for the problem.

b) In order to draw the graphs of inequalities, let us convert the inequalities to equations. Therefore, we have

$$6x_1 + 2x_2 = 36 \text{ or } 3x_1 + x_2 = 18$$

$$3x_1 + 5x_2 = 30$$

$$x_1 + 4x_2 = 20$$

Using the simple technique for drawing graphs we arrive at the following diagram that gives the feasible region.

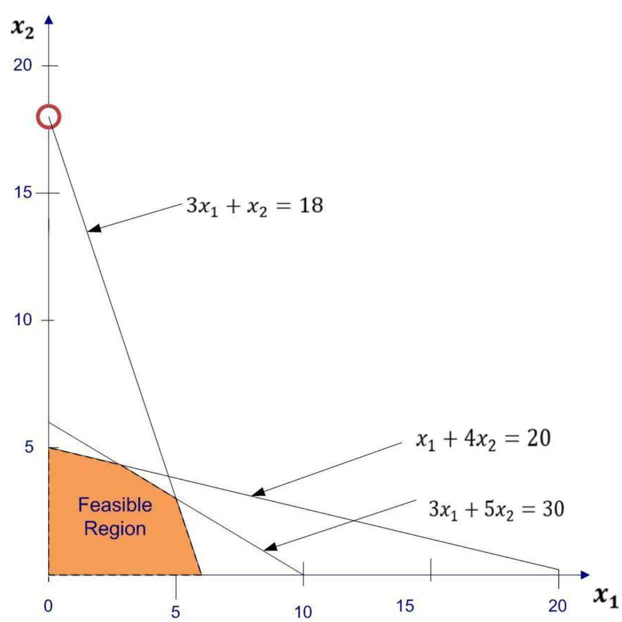

Fig. 26.1: Graph of constraints and Feasible region

To obtain the optimum solution, we transform the objective function and draw the graph of the objective function to see at which point (basic feasible solution) it is tangent.

Thus, we have

$$8x_2 = \pi - 10x_1$$

$$or, \qquad x_2 = \frac{\pi}{8} - \frac{10}{8}x_1$$

$$or, \qquad x_2 = \frac{\pi}{8} - \frac{5}{4}x_1$$

The optimum solution is shown in Fig. 26.2

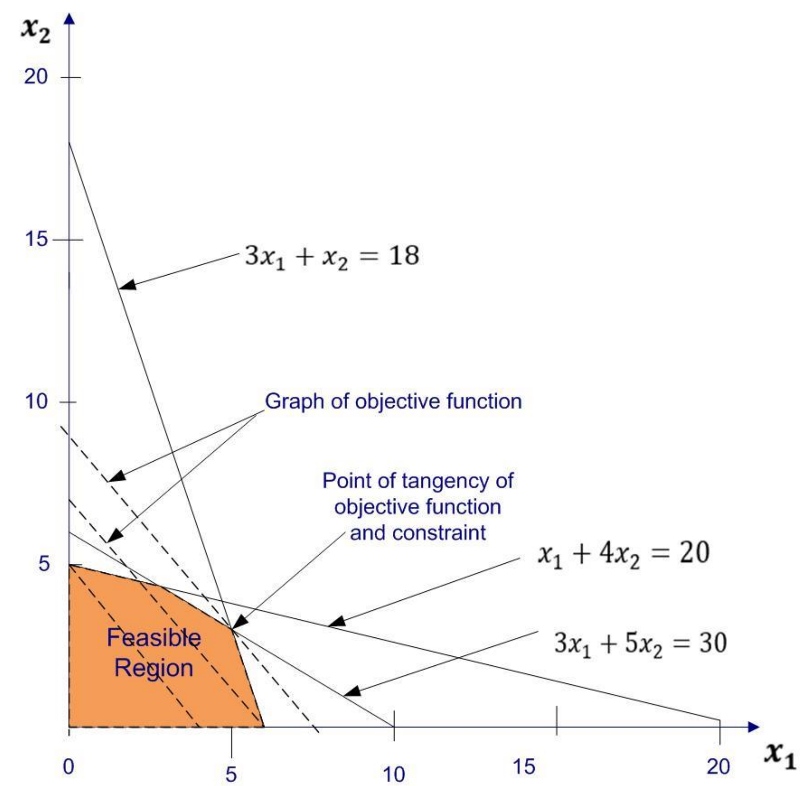

Fig. 26.2: Optimum Solution for a Maximization problem

Now, to find the co-ordinates of the point of tangency, we have to solve the equations that intersect at that point. It is seen from Fig 26.2, that the point of tangency of objective function and basic feasible solution is given by the intersection of the equation, $3x_1 + x_2 = 18$ and

$3x_1 + 5x_2 = 30$. Solving the two equation gives $x_1 = 5 \; and \; x_2 = 3$

Therefore, Tata Steel should produce $5 \; units \; of \; x_1 \; and \; 3 \; units \; x_2$ to maximize its profit.

Alternative Solution to find the Optimum Solution:

The co-ordinates of the corner points or the basic feasible solutions are: (0,5), (2.8, 4.2), (5,3) and (6,0)

Now putting the values in the profit function gives:

$\pi = 10(0) + 8(5) = 40$

$\pi = 10(2.8) + 8(4.2) = 61.6$

$\pi = 10(5) + 8(3) = 74$

$\pi = 10(6) + 8(0) = 60$

Comparing the value of profit, it is clearly seen that the profit is maximum for $x_1 = 5\ and\ x_2 = 3$

c) The maximum profit is equal to $\pi = 10(5) + 8(3) = 74$ or ₹ 74000

26.2. Minimization Problem

Suppose the daily nutrient requirement for you to stay healthy is 30 grams of carbohydrates, 20 grams of protein and 24 grams of dietary fibre. Your dietician recommends two foods $p\ and\ q$. A kilogram of Food p contains 2 grams of carbohydrate, 5 grams of protein and 2 grams of dietary fibre. Similarly, a kilogram of Food q contains 6 grams of carbohydrate, one gram of protein and 3 grams of dietary fibre. The price per kilogram of Food $p_1\ and\ q_1$ is ₹80 and ₹160 respectively.

a) Formulate a mathematical model for the given information.

b) How much of $p_1\ and\ q_1$ would you purchase so that you spend as less as possible and at the same time stay healthy?

c) What is the minimum amount you would have to pay?

Solution:

a) Let us first represent the given information in a tabular form:

Name of nutrient	**Nutrient content per unit of Food (in gram)**		**Minimum daily requirement of each nutrient (in gram)**
	$\boldsymbol{p}$	$\boldsymbol{q}$	
Carbohydrate	2	6	30
Protein	5	1	20
Dietary fibre	2	3	24
Cost per unit of food (in ₹)	80	160	

Clearly, your objective is to mimimize the cost of diet. Using simple mathematical function between the variables, the objective function and the constraints may be written in mathematical language as

$$Minimize$$

$$C = 80p + 160q \qquad C \text{ is the Total Cost}$$

$$subject\ to$$

$$2p + 6q \geq 30$$

$$5p + q \geq 20$$

$$2p + 3q \geq 24$$

The model just constructed is a linear programming problem with inequality constraints. The graphical analysis for solving the problem requires us to draw the graphs of the constraints and find the feasible region and then arrive at the solution for the problem.

b) In order to draw the graphs of inequalities, let us equate the inequalities to zero. Therefore, we have

$$2p + 6q = 30 \quad or, \qquad p + 3q = 15$$

$$5p + q = 20$$

$$2p + 3q = 24$$

Using the simple technique for drawing graphs we arrive at the following diagram that gives the feasible region.

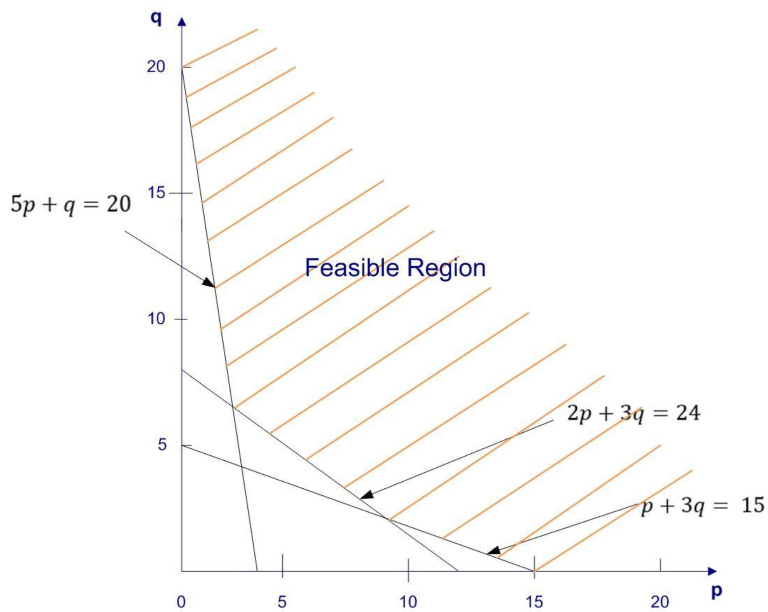

Fig. 26.3: Feasible region for diet problem

To obtain the optimum solution, we transform the objective function and draw the graph of the objective function to see at which point (basic feasible solution) it is tangent.

Thus, we have

$$q = \frac{C}{160} - \frac{80}{160}p$$

$$or, \qquad q = \frac{C}{160} - \frac{1}{2}p$$

The optimum solution is shown in Fig. 26.4

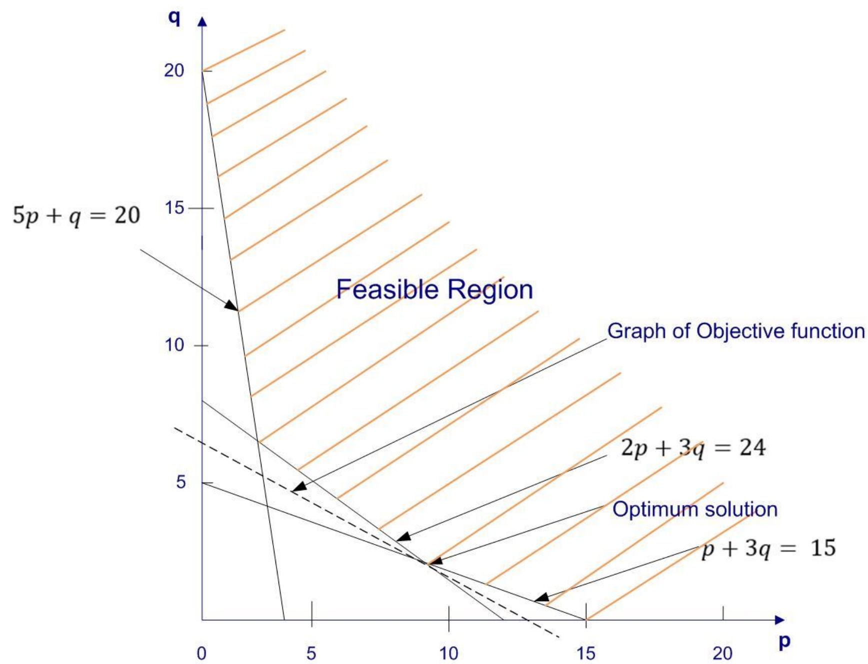

Fig. 26.4: Optimum solution for a Minimization Problem

To find the co-ordinates of the point of tangency, we have to solve the equations that intersect at that point. It is seen from Fig 26.4, that the point of tangency of objective function and basic feasible solution is given by the intersection of the equation, $2p + 3q = 24$ and $p + 3q = 15$ Solving the two equation gives $p = 9 \; and \; q = 2$

Therefore, you have to purchase $9 \; kg \; of \; Food \; p \; and \; 2 \; kg \; of \; Food \; q$ to meet the minimum daily requirement of nutrients.

c) The minimum cost is equal to $\mathrm{C} = 80(9) + 160(2) =$ ₹1040

